LARGE DEVIATION FUNCTIONS OF TIME-INTEGRATED CURRENT IN STOCHASTIC TRAFFIC MODELS

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ABSTRACT

Here we study large deviation functions (LDF) of time-integrated current for a number of variants of TASEP by adapting the iterative Measurement-and-Feedback method [1]. Firstly, we study LDF for TASEP on a ring, both analytically and numerically to justify this method. We compare our exact results using Bethe anstaz with simulations based on the iterative method, analyzing order of corrections in this method and derive conditions of its applicability. Then, we adapt the existing methods to the setting of discrete time Markov chains to study LDF for discrete time TASEP with open boundaries as a minimal stochastic traffic model.

INTRODUCTION

A one-dimensional lattice system of L sites, each site being either occupied by one particle or empty. At each time step, particles jump to the right with probability p, provided that site is empty.

1. Open boundaries condition: particles can enter and exit the system from boundaries.
2. Periodic (ring) boundaries condition: number of particles is fixed.

- Time integrated current, denoted by Q_T is defined as normalized total number of bulk jumps during time interval $[0,T]$.

$$Q_T = \frac{1}{T}\sum_{i=0}^{T-1} J_{C_i C_{i+1}} \tag{1}$$

- Q_T obeys large deviation principle (LDP) in long time limit [2], i.e. $\lim_{T\to\infty}\mathbb{P}(Q_T = q) = e^{-T\mathcal{F}(q)}$.
- $\mathcal{F}(q)$ is called LDF of Q_T. Gartner-Ellis theorem [2] states that $\mathcal{F}(q)$ is Legendre transform of scaled cumulant generating function of Q_T, i.e. $\mathcal{F}(q) = \sup_{s\in\mathbb{R}}\{sq - G(s)\}$.

BETHE ANSTAZ FOR TASEP ON A RING FOR $s < 0$

Each configuration of the system is represented by a strictly increasing sequence of integers of length N with elements from $\{1, 2\cdots N\}$ which determined the positions of N particles. i.e. $C = \{n_1, n_2, \cdots, n_N\}$. The eigenfunction of a configuration C is written as

$$\psi(n_1, n_2\cdots n_N) = \sum_{\sigma\in S_N} A_\sigma z_{\sigma(1)}^{n_1} z_{\sigma(2)}^{n_2}\cdots z_{\sigma(N)}^{n_N} \tag{2}$$

where S_N is a set of all permutations of the integers $1, 2, \cdots, N$.
The Bethe equations are

$$z_k^L = \prod_{i=1}^{N} -\frac{e^s - z_k}{e^s - z_i} \quad \text{for } k = 1, \cdots, N \tag{3}$$

For any solution $\{z_k\}$, (2) gives an eigenvector of matrix $\tilde{U}$ with eigenvalue $\Lambda = e^s(\sum_{k=1}^{N}\frac{1}{z_k}) - N$. From periodicity condition

$$A_{\sigma(1),\sigma(2),\cdots,\sigma(N)} = A_{\sigma(2),\sigma(3),\cdots\sigma(N),\sigma(1)} z_{\sigma(1)}^L \tag{4}$$

Our starting point is the results in [3] where the authors estimated $\{z_k\}$ for which Λ is maximized. It is the case where $N-1$ of $z_k \approx e^s$, and one is $e^{(1-N)s}$ (Here we assume z_1 is this one).

- The non-zero terms in equation (2) are those in which the amplitudes can be written in terms of the amplitude of the identity permutation, using the periodicity condition.
- There are N nonzero terms; all transpositions of 1 with other integers.
- By substitution we obtain

$$\psi(n_1, n_2\cdots n_N) = \sum_{j=1}^{j=N} e^{s[L(j-1)-\sum_{k=1}^{N}(n_j - n_k)]} \tag{5}$$

- The equation (5) is translational invariance.
- $\psi\{n_1, \cdots n_N\}$ attains minimums at equidistant configurations, and maximum at configurations with one cluster.

MODELS

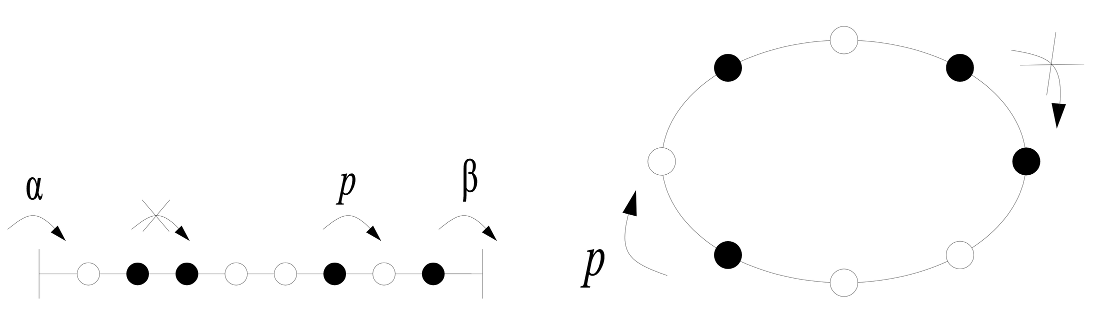

Figure 1: Open boundaries (left); Periodic boundaries (right)

ERROR OF THE ESTIMATION

To evaluate this, we calculate the ratio between the largest and second largest values of $\psi(C)$. The latter attains in configurations with two clusters, one is of size $N-1$, and the other one is a single particle cluster. c is the distance between the clusters.

$$\frac{\psi\{n, n+1, \cdots, n+N-2, n+N-1+c\}}{\psi\{n, n+1, \cdots, n+N-1\}} = e^{cs} + O(e^{s(N-1)(L-N)}) \tag{12}$$

RESULTS

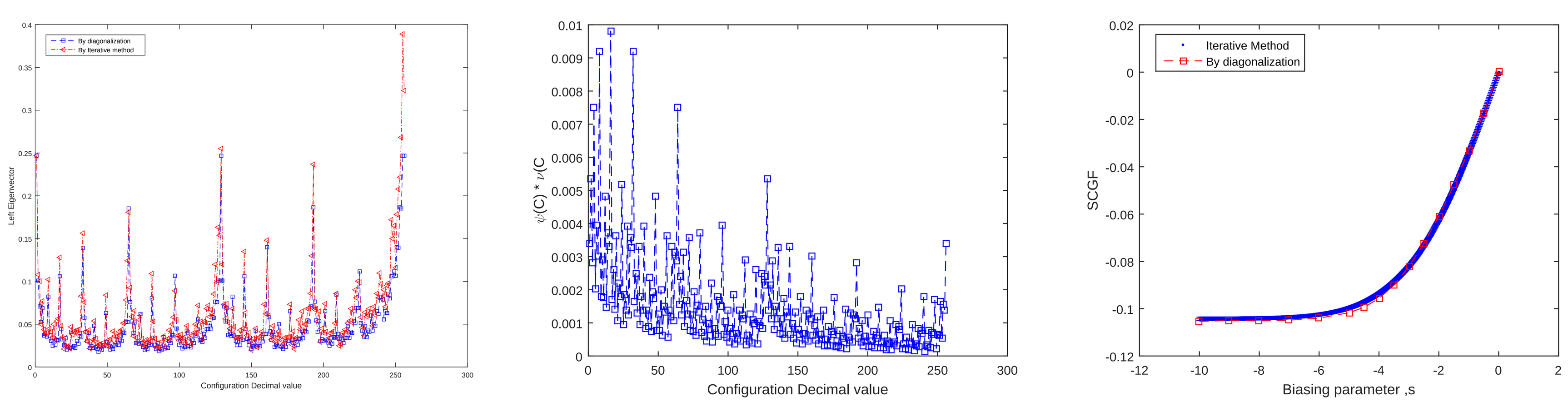

Figure 1 (left); Left eigenvector corresponding to the largest eigenvalue of $\tilde{U}$ from diagonalized $\tilde{U}$, and simulation. (middle); Product of right and left eigenvector corresponding to the largest eigenvalue of $\tilde{U}$ from diagonalized $\tilde{U}$, for discrete time TASEP with open boundaries condition, $L = 8$, $\alpha = 0.8$, $\beta = 0.8$ and $s = -2$. (right); Scaled cumulant generating function for discrete time TASEP with open boundaries condition, $L = 8$, $\alpha = 0.8$, $\beta = 0.8$ from diagonalized $\tilde{U}$, and simulation.

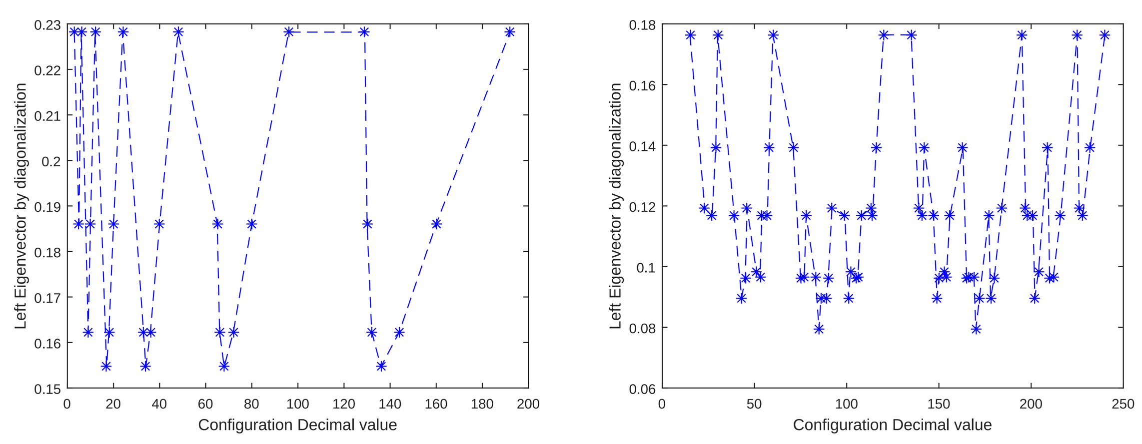

Figure 1 Left eigenvector corresponding to the largest eigenvalue of tilted operator of continuous time TASEP on a ring with $L = 8$, and and $s = -2$ (left); Low density $\rho = 0.25$. (left); half-filling

ITERATIVE METHOD [1]

We adapt the iterative method [1] to discrete time TASEP. Let, $U = \{u(C, C')\}$ represent the transition matrix of the system. Define the tilted transition matrix by

$$\tilde{u}(C, C') = \begin{cases} e^{sJ_{CC'}}u(C, C') & C \neq C' \\ 1 - \sum_{C'\neq C} u(C, C') & C = C' \end{cases} \tag{6}$$

The main idea of the iterative method [1] is to create a physical system corresponding to the biased process so that a rare event in the original system is a typical event in the new system.
Structure of transition matrix of the auxiliary system for discrete TASEP is

$$u_{C,C'}^{\text{aux}} = A_s e^{sJ_{CC'}} u(C, C')\frac{\psi(C')}{\psi(C)} \tag{7}$$

where A_s is a normalization constant, and $\psi(C)$ are entries of the left eigenvector corresponding to $\mu(s)$. Steps of the iterative method are as follows [1]:

- Measure $\langle e^{\delta_s \tau Q_\tau}\rangle_C$ as a function of C in the original system.
- Then, depending on the value of $\langle e^{\delta_s \tau Q_\tau}\rangle_{C'}$ we modify the transition probability to

$$u^{\delta_s}(C, C') = A_{\delta_s} u(C, C') e^{\delta_s J_{CC'}} \frac{\langle e^{\delta_s \tau Q_\tau}\rangle_{C'}}{\langle e^{\delta_s \tau Q_\tau}\rangle_C} \tag{8}$$

- Next, in the modified system, we measure the expected value of the same quantity $e^{\delta_s \tau Q_\tau}$, denoted by $\langle e^{\delta_s \tau Q_\tau}\rangle_C^{\delta_s}$.
- Again, we define the second modified transition probability as

$$u^{2\delta_s}(C, C') = A_{2\delta_s} u^{\delta_s}(C, C') e^{\delta_s J_{CC'}} \frac{\langle e^{\delta_s \tau Q_\tau}\rangle_{C'}^{\delta_s}}{\langle e^{\delta_s \tau Q_\tau}\rangle_C^{\delta_s}} \tag{9}$$

- We iterate this procedure for many times. Then, we obtain a set of transition probabilities

$$u^{l\delta_s}(C, C') = A_{l\delta_s} u(C, C') e^{l\delta_s J_{CC'}} \prod_{i=0}^{l-1} \frac{\langle e^{\delta_s \tau Q_\tau}\rangle_{C'}^{i\delta_s}}{\langle e^{\delta_s \tau Q_\tau}\rangle_C^{i\delta_s}} \tag{10}$$

with $l = 0, 1, 2, \cdots$.
- The iterative method is based on $\langle Q_T\rangle^s \approx \frac{\langle Q_T e^{sTQ_T}\rangle}{\langle e^{sTQ_T}\rangle}$
- From the formula, we obtain the expected value of any quantity in the biased system. For example, for the LDF of Q_T

$$\mathcal{F}(q) = \sup_s\, [sq - \sum_{l=0}^{l=M-1} \langle Q_T\rangle^{l\delta_s}\,\delta_s] + O(\delta_s^2) \tag{11}$$

with $s = M\delta_s$.

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FUTURE RESEARCH

This method will be applied in future to the Nagel-Schreckenberg model, which serves as a minimal discrete model of freeway traffic.

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