Turbulence: A big unresolved puzzle in classical mechanics

Universality class of fully developed turbulence

- conservation law
- cascade in the inertial range

\[ E = \frac{1}{2} \int \mathbf{v}^2 \, dx : \text{energy (2D, 3D)} \]

\[ \Omega = \frac{1}{2} \int d^2 \mathbf{x} (\nabla \times \mathbf{v})^2 : \text{enstrophy (2D)} \]

\[ H = \int d^3 \mathbf{x} \, \mathbf{v} \cdot (\nabla \times \mathbf{v}) : \text{helicity (3D)} \]

Quantum turbulence: new turbulent system with quantized vortices

model equation: nonlinear Schrödinger equation (NLSE)

\[ i \hbar \frac{\delta H}{\delta \psi^*} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + g(|\psi|^2 - \bar{\rho}) \right] \psi \]

\[ H = \int d^3 x \left[ \frac{\hbar^2}{2M} \nabla \psi^* \nabla \psi + \frac{g}{2} (|\psi|^2 - \bar{\rho})^2 \right] \]

\[ \rho = M |\psi|^2 \quad \bar{\rho} = \Re \psi^* \nabla \psi \]

Quantum fluid equation

\[ \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \]

Quantized vortex

\[ \rho = \frac{\hbar}{\mathcal{M}} \]

Turbulence simulation

\[ (i - \gamma) \hbar (\partial_t - v_{\text{ext}} \cdot \nabla) \psi = \frac{\delta H}{\delta \psi^*} \]

\[ v_{\text{ext}} : \text{external current (divergence free)} \]

Universality class with the helicity cascade has not been observed!

Quantum universality class

Abelian turbulence

non-Abelian turbulence

Extension to non-Abelian vortex: spin degrees of freedom

from scalar to spinor wave function

\[ \psi \rightarrow \begin{pmatrix} \psi_2 \\ \psi_1 \\ \psi_0 \\ \psi_{-1} \\ \psi_{-2} \end{pmatrix}^T \]

for spin-2 case

cyclic ground state for \( g_{\text{int}} > 0 \)

Abelian collision: topology is not conserved

non-Abelian collision: topology is conserved

Helicity in quantum fluid

\[ H = H_{\text{twist}} + H_{\text{wrtb}} = H_{\text{linking}} + H_{\text{vrtx}} + H_{\text{wrtb}} \]

Abelian knot

non-Abelian knot

H_{\text{linking}} discretely changes through vortex reconnections → helicity is not conserving quantity

H_{\text{linking}} is conserved because reconnection is prohibited → topologically protected linking number cascade

Conclusion: We first obtain a new universality class \( \nu = 2/3 \quad \eta = 7/3 \) in non-Abelian quantum turbulence with topologically protected linking number cascade